INVERSE SPECTRAL PROBLEMS USED FOR THE SYNTHESIS
OF DIFFUSIONAL LIGHT GUIDES

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The method of the inverse problem of scattering theory is used to analyze processes of radiation propagation and mass transfer in optical guiding systems.

In investigations of fiber-optic communication lines and elements of integral optics, much attention is paid to problems of the synthesis of these devices, having several aspects for consideration. First, one must establish the law of distribution of the main characteristics, such as the permittivity ε and the magnetic permeability μ of the synthesized optical devices, determining their optical transmission characteristics. Second, one must develop the optimum regimes for the processes of formation of the assigned distributions of ε and μ in these devices.

If diffusional waveguides are investigated, in particular, then the successive formulation and analysis of two types of inverse problems are required: electrodynamic and heat- and mass-exchange problems. The first come down to inverse spectral problems of quantum mechanics for the Schrödinger equation [1, 2], with which the nonuniform distributions of the permittivity and permeability of the waveguide are reconstructed from the spectral characteristics. In the second formulation, the technological process of nonuniform distribution of the permittivity ε and permeability μ obtained in the solution of the first problem is modeled using the nonsteady diffusion equation describing the variation of the concentration u(x, t) of the admixture. The present article is devoted to an investigation of the laws connecting the diffusional processes of synthesis of waveguides with their optical properties.

Wave propagation in guiding systems, such as plane stratified media and optical waveguides, is described by scalar wave equations derived from the Maxwell equations. The transverse component of the electric field of a light wave propagating along the z axis of a regular waveguide can be represented as

$$E(x,\ z,\ t) = F(k,\ x,\ \theta)\exp\left[-ik(\sin\theta z - t/c)\right], \tag{1}$$

where $k = \omega/c$ is the wave number of free propagation; ω is the frequency; c is the speed of light. In this case with $\mu = 1$ the wave equation is converted into a one-dimensional equation,

$$(d^2/dx^2)F(k,\ \theta,\ x) + k^2[\varepsilon(x) - \sin^2\theta]F(k,\ \theta,\ x) = 0, \tag{2}$$

which, both for a fixed k and for a fixed angle of inclination to the axis of the waveguide $(\theta' = 90 - \theta)$, is reduced to the Schrödinger equation

$$-(d^2/dx^2)F(\beta,\ x) + V(x)F(\beta,\ x) = EF(\beta,\ x), \tag{3}$$

where $\beta = k\cos\theta$ and $E = \beta^2$.

In the first case, when the values of θ are varied and k = const, $V(x) = [1 - \varepsilon(x)]k^2$. In the second case, with variable values of k and θ = const, (2) is converted into (3) through a Liouville transformations: using the substitutions $d\xi/dx = q^2(x)$, $q(x) = [\varepsilon(x) - \sin^2\theta]^{1/4}$, and $V(\xi) = q^{-1}\dfrac{\partial^2 q}{\partial\xi^2}$ and the change of the variable x to ξ.

The asymptotic behavior of the function $F(\beta, x)$ is determined by the relations

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$$\lim_{x\to\infty} F(\beta, x) = \exp(-i\beta x) + R(k, \theta)\exp(i\beta x) \tag{4}$$

for mod β of the continuous spectrum [R(k, θ) is the reflection coefficient] and

$$\lim_{x\to\infty} F(\beta_n, x) = M_n \exp(-\beta_n x) \tag{4'}$$

for mod β_n of the discrete spectrum, where M_n^2 are normalization constants:

$$M_n^2 = \left[\int_0^\infty F^2(\beta_n, x)\,dx\right]^{-1}.$$

In the direct problem of the quantum theory of scattering, the assignment of V(x) determines the solution F(β, x) and the scattering characteristics R(β) and $\{\beta_n, M_n\}$. In the inverse problem V(x) and the solution are reconstructed from the scattering characteristics. The basic equations of the inverse scattering problem are the Gel'fand–Levitan and Marchenko equations,

$$K(x, x') + Q(x, x') + \int_x^\infty K(x, y)Q(y, x')\,dy = 0. \tag{5}$$

$$K^{GL}(x, x') + Q^{GL}(x, x') + \int_0^x K^{GL}(x, y)\,Q^{GL}(y, x')\,dy = 0. \tag{5'}$$

They are solved uniquely for K(x, y) or $K^{GL}(x, y)$. Serving as the initial data for the solutions of (5) and (5') are scattering data or the spectral characteristics, determining the kernel Q(x, y) of the Marchenko equation (5) and $Q^{GL}(x, y)$ of the Gel'fand–Levitan equation (5'):

$$Q(x, y) = \frac{1}{\pi}\int_0^\infty R(\beta)\exp i\beta(x+y)\,d\beta + \sum_{n=1}^{N} M_n^2 \exp[-\beta_n(x+y)], \tag{6}$$

$$Q^{GL}(x, y) = \int_0^\infty \mathring{\varphi}(\beta, x)\,\mathring{\varphi}(\beta, y)\,d[\rho(\beta) - \mathring{\rho}(\beta)] + \sum_{n=1}^{N} C_n^2\,\mathring{\varphi}(\beta_n, x)\,\mathring{\varphi}(\beta_n, y). \tag{6'}$$

The functions K, in turn, are connected with the distribution V(x) being sought in a simple relation

$$V(x) = \mathring{V}(x) - 2\frac{d}{dx}K(x, x) \tag{7}$$

and by Jost's solutions* of Eq. (3), determined by the conditions

$$\lim_{x\to\infty} f_\pm(\beta, x) = \exp(\pm i\beta x),$$

$$f_\pm(\beta, x) = \exp(\pm i\beta x) + \int_x^\infty K(x, x')\exp(\pm i\beta x')\,dx', \tag{8}$$

*Regular solutions $\phi(\beta, x)$, satisfying the boundary conditions $\varphi(\beta, x=0) = \mathring{\varphi}(\beta, x=0) = 0$ and $\varphi'(\beta, x=0) = \mathring{\varphi}'(\beta, x=0) = 1$, are determined by the relation

$$\varphi(\beta, x) = \mathring{\varphi}(\beta, x) + \int_0^x K^{GL}(x, x')\,\mathring{\varphi}(\beta, x')\,dx'; \tag{8'}$$

in (6') $C_n^2 = \left[\int_0^\infty \varphi^2(\beta_n, x)\,dx\right]^{-1}$ and $\rho(\beta)$ is the spectral measure (see [5]).

from which, in accordance with the condition (4), the physical solution can be written at once:

$$F(\beta, x) = f_-(\beta, x) + R(\beta) f_+(\beta, x),$$
$$F(\beta_n, x) = M_n f_+(i\beta_n, x). \tag{9}$$

The limits of integration in (5), (5'), (8), and (8') and the means of assigning the spectral information depend on the formalism used for the inverse problem. Equations (5)-(9) are written by the Marchenko method and can be used in the analysis of regular cylindrical waveguides or plane inhomogeneous media at $x > 0$ if $\varepsilon(x) = 1$ and $\mu(x) = 1$ at $x \leq 0$.

For a plane waveguide it is convenient to use the inverse problem over the entire axis $-\infty < x < \infty$ [3], a peculiarity of which consists in the fact that two equations must be analyzed instead of the one Marchenko equation (5). However, by virtue of the fact that $V(x)$ is expressed both through the kernel $K_1(x, y)$ of one integral equation and through the kernel $K_2(x, y)$ of the other, it is sufficient to find one of the $K(x, y)$ from an equation coinciding with (5).

If one is investigating a situation when not only $\varepsilon(x) \neq 1$ but also $\mu(x) \neq 1$, then instead of (2) one must consider the equation

$$\frac{d^2}{dx^2}\Phi(k, \theta, x) + k^2[\varepsilon(x)\mu(x) - \sin^2\theta]\Phi(k, \theta, x) = \frac{d}{dx}\ln\mu(x)\frac{d}{dx}\Phi(k, \theta, x),$$

which is again reduced to (3) using a Liouville transformations. Now, however, twice as much spectral information is needed to determine $\varepsilon(x)$ and $\mu(x)$: for two angles θ and all k [4].

In a number of cases, when $R(\beta)$ is a fractional-rational function of β or reflection is altogether absent, $R(\beta) = 0$ (see, e.g., [5]), the spectral kernel $Q(x, y)$ can be factored [written in the form of a finite sum of products $\sum_{n=1}^{M}\varphi_n(x)\chi_n(y)$] and the integral equation (5) changes into an algebraic system of a finite number of equations for $K(x, y)$. As a result, we can find $V(x)$ and hence $\varepsilon(x)$ and the solution, in explicit form using Eqs. (7)-(9). We use the technique of degenerate kernels to obtain the permittivity profile of the guiding systems providing the assigned multimode regime of propagation of the electric field of the light wave [6]. Let $R(\beta) = 0$, which corresponds to the nonreflective case of the problem over the entire axis. Then only the contribution from states of the discrete spectrum remains in $Q(x, y)$:

$$Q(x, y) = \sum_n^N M_n^2 \mathring{f}(i\beta_n, x)\mathring{f}(i\beta_n, y)\Big|_{\mathring{\varepsilon}(x)=1} =$$
$$= \sum_n^N M_n^2 \exp[-\beta_n(x+y)]. \tag{10}$$

Similarly, for $K(x, y)$ we have

$$K(x, y) = -\sum_n^N M_n^2 f(i\beta_n, x)\mathring{f}(i\beta_n, y)\Big|_{\mathring{\varepsilon}(x)=1} = -\sum_n^N M_n^2 f(i\beta_n, x)\exp(-\beta_n, y). \tag{11}$$

For $f(i\beta_n, x)$ we obtain from (5) the system of algebraic equations

$$f(i\beta_n, x) = \sum_j^N \mathring{f}(i\beta_j, x)[P^{-1}(x)]_{jn} \tag{12}$$

with the matrix of coefficients $P_{nj}(x)$

$$P_{nj}(x) = \delta_{nj} + \int_x^\infty \mathring{f}(i\beta_n, x')M_n^2\mathring{f}(i\beta_j, x')\,dx'\Big|_{\mathring{\varepsilon}(x)=1} = \delta_{nj} + \frac{M_n^2\exp[-(\beta_n+\beta_j)x]}{\beta_n+\beta_j}. \tag{13}$$

Using (11), (7) and (8), we obtain

$$V(x) - \overset{\circ}{V}(x) = k^2[\overset{\circ}{\varepsilon}(x) - \varepsilon(x)] = -2\frac{d^2}{dx^2}\ln \mathrm{Det}\,\|P_{nj}(x)\|. \tag{14}$$

In this case the solution of Eq. (3) is written in closed form,

$$f(\beta, x) = \overset{\circ}{f}(\beta, x) + \sum_{n,j}^{N} \overset{\circ}{f}(i\beta_n, x)M_n^2[P^{-1}(x)]_{nj}\int_x^\infty \overset{\circ}{f}(i\beta_j, x')\overset{\circ}{f}(\beta, x')\,dx', \tag{15}$$

where $\overset{\circ}{f}(\beta, x) = \exp(i\beta x)$ for $\overset{\circ}{\varepsilon}(x) = 1$, while in the general case it is a function that is a solution of (3) with a certain $\overset{\circ}{\varepsilon}(x)$ known in advance [7]. In the particular case of one mode with the parameters β_1 and M_1, the permittivity is

$$\varepsilon(x) = 1 + \frac{2\beta_1}{k^2\,\mathrm{ch}^2\,2\beta_1(x - x_0)}, \tag{16}$$

where $x_0 = (2\beta_1)^{-1}\ln(2\beta_1/M_1^2)$. We write the corresponding solution of Eq. (3):

$$f(\beta, x) = \exp(i\beta x)\left[1 + \frac{\beta_1\exp[-\beta_1(x - x_0)]}{(\beta_1 - i\beta)\mathrm{ch}[\beta_1(x - x_0)]}\right]. \tag{17}$$

In an analysis of cylindrical waveguides or plane media for which $\varepsilon(x) = \mu(x) \equiv 1$ at $x \leq 0$, in $Q(x, t)$ of (6) one must take into account, besides the sum over states of the discrete spectrum, an integral term, which can be replaced by a sum of residues at points of bound states. We shall not dwell on the investigtion of this case here. We only note that it would be more convenient to use the Gel'fand–Levitan method [5], since $\rho(\beta) = \overset{\circ}{\rho}(\beta)$ and the integral part of $Q^{GL}(x, y)$ of (6') makes no contribution.

Let us consider a more complicated situation, generalizing the preceding ones, when along with the directional modes of the discrete spectrum there are also emission modes. The scattering function $S(\beta)$ remains fractional-rational in this case, however:

$$S(\beta) = \prod_j^N \frac{\beta + i\beta_j}{\beta - i\beta_j}\,\frac{\beta + ia_j}{\beta - ia_j}. \tag{18}$$

We find $Q(x, y)$ from Eq. (6) with $R(\beta) = 1 - S(\beta)$. Closing the integration contour in the upper half-plane β, using the residue theorem we obtain

$$Q(x, y) = \sum_p^N A_p\exp[-d_p(x + y)]. \tag{19}$$

The contributions from both the bound states ($d_p = i\beta_p$, $\mathrm{Re}\,\beta_p = 0$) and other bands $S(\beta)$ at the points $d_p = a_p$ are now combined in this expression. In the general case a_p are complex and are distributed symmetrically with respect to the imaginary axis. These bands can correspond to "resonance" modes – weakly damped modes of the continuous spectrum.

Thus,

$$A_p = \frac{2\beta_p(\beta_p + a_p)}{(\beta_p - a_p)}\prod_{n\neq p}^{N_b}\frac{(\beta_p + a_n)(\beta_p + \beta_n)}{(\beta_p - \beta_n)(\beta_p - a_n)} + M_p^2 \quad \text{for } p = 1, \ldots, N_b,$$

$$A_p = \frac{2a_p(\beta_p + a_p)}{(a_p - \beta_p)}\prod_{n\neq p}^{N}\frac{(a_n + a_p)(\beta_n + a_p)}{(a_p - a_n)(a_p - \beta_n)} \quad \text{for } p = N_b + 1, \ldots, N.$$

Substituting Q in the form (19) into the Marchenko equation (5), we obtain

$$K(x, y) = -\sum_j^N A_j f(id_j, x)\exp(-d_j y). \tag{20}$$

The relations (14) and (15) remain valid for the potential and the solutions if M_n^2 is replaced by A_n and P_{ni} are determined not from (13) but from

$$P_{ni}(x) = \delta_{ni} + A_n \exp[-(d_n + d_i)x]/(d_n + d_i). \tag{21}$$

By varying the normalization of the bound states M_p^2, we obtain N_b, a parametric family of phase-equivalent media. If $M_p^2 = -i\mathrm{Res}\, R(\beta)$ for $\beta = i\beta_p$, then all $A_p = 0$ for $p = 1, \ldots, N_b$ and only the sum over $(N - N_b)$ states remains in Eqs. (20), (19), etc. The $\varepsilon(x)$ and $f(\beta, x)$ obtained for such M_p^2 can be used as reference values for determining the $\varepsilon(x)$ and $f(\beta, x)$ corresponding to the arbitrary normalizations of M_p^2. This way of reconstructing $\varepsilon(x)$ sometimes proves more convenient than the preceding one, since it operates with systems of algebraic equations of lower order. So, we can write the distribution $\varepsilon(x)$ [and hence $n(x)$ also, since $\varepsilon(x) = n^2(x)$] and the corresponding solutions for the given regime of operation of the light guide in explicit form.

Inverse Problem for the Diffusion Equation Modeling the Process of Formation of a Fiber

Now let us consider the mathematical model for the technological process of creation of the permittivity distribution $\varepsilon(x)$ found above (14) of a light guide for multimode or one-mode light propagation. To describe the diffusion process we use the system of equations and boundary conditions

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial}{\partial x}\left[D(x)\frac{\partial u(x, t)}{\partial x}\right], \tag{22}$$

$$u(x, T) = \mathrm{const}\,\Delta\varepsilon(x) = \mathrm{const}\,[\varepsilon(x) - \mathring{\varepsilon}(x)], \tag{23}$$

$$u'(0, t) = 0, \quad u(a, t) = 0, \tag{24}$$

where a is an arbitrary fixed value of x and, in general, $a \to \infty$.

The condition (23) is written under the assumption that for a low concentration of diffusant molecules $u(x, t) \sim \Delta\varepsilon(x, t)$. This condition requires that the concentration distribution $u(x, t)$ of the admixture by the time $t = T$ leads to the $\Delta\varepsilon(x)$ assigned above (14) as a result of the diffusion process. The boundary condition (24) formulates the requirement of symmetry of the distribution of the admixture.

We first consider the case when $D(x)$ is a certain known function. We represent the unknown concentration distribution of the admixture in the form of an expansion in a complete system of orthonormal functions $\chi(p_m, x)$,

$$u(x, t) = \sum_m^{\infty} \varphi_m \exp[-p_m^2(t - T)]\chi(p_m, x) \equiv \sum_m^{\infty} A_m(t)\chi(p_m, x), \tag{25}$$

where the coefficients of the expansion are

$$A_m(t) = \varphi_m \exp[-p_m^2(t - T)] \equiv A_m(T)\exp[-p_m^2(t - T)], \quad 0 \leqslant t \leqslant T, \tag{26}$$

since $A_m(T) = \varphi_m$, as is easy to see from (25). The coefficients φ_m are determined, in turn, from the relation

$$\varphi_m = \int_0^a dx\, \Delta\varepsilon(x)\chi(p_m, x)\,dx, \tag{27}$$

following from (25) with $t = T$, (23), and the orthogonality of the functions $\chi(p_m, x)$. After substituting (25) into (22)-(24) we arrive at the classic Sturm–Liouville problem for the functions $\chi(p_m, x)$,

$$\frac{d}{dx}\left[D(x)\frac{d}{dx}\chi(p_m, x)\right] = -p_m^2\chi(p_m, x); \tag{28}$$

$$\chi'(p_m, 0) = 0, \ \chi(p_m, a) = 0. \tag{29}$$

Finding the spectrum of eigenvalues and eigenfunctions of χ for the problem (28), (29), having solved the direct problem, from Eq. (27) we obtain ϕ_m, while we also determine u(x, t) at any time t using (25). The value u(x = 0, t = 0) is the concentration of the admixture which must be assigned at the initial time t and at the point x = 0 in order to obtain the distribution $\Delta\varepsilon(x)$ assigned by (16) or (14) by the time t = T for the known function D(x). And this will be the boundary condition for control of the diffusion process.

Now we consider the situation when the diffusion function D(x) is not known in advance, but either u(x = 0, t) or its derivative is measured experimentally at any time t. Then one can formulate the inverse problem of determining the functions D(x) and solutions u(x, t) of the diffusion equation (22). The fundamental possibility of solving such a problem for Eq. (22) with a piecewise-constant function D(x) and the initial condition u(x, t = 0) = 0 was shown by A. M. Denisov [8]. Let u'(0, t) be known. We shall investigate the problem (22), (23), (24) with the boundary conditions

$$u(0, t) = 0, \tag{30}$$

$$u(a, t) + BD_a \left.\frac{\partial u(x, t)}{\partial x}\right|_{x=a} = \mu(t). \tag{31}$$

We note that we can alter the formulation when necessary, taking u'(0, t) = 0 or interchanging the conditions at zero and at the point a. Using a generalized Laplace transformation, we represent u(x, t) in the form

$$u(x, t) = \int_0^\infty \varphi(p)\chi(p, x)\exp[-p^2(t-T)]\,dp. \tag{32}$$

At t = T

$$u(x, T) = \text{const}\,\Delta\varepsilon(x) = \int_0^\infty \varphi(p)\chi(p, x)\,dp = \int_0^\infty dp\,\psi(p, x). \tag{33}$$

We write the equation and boundary conditions for $\psi(p, x)$, using (22), (30), (31), and an inverse Laplace transformation:

$$\frac{d}{dx}\left[D(x)\frac{d}{dx}\psi(p, x)\right] = -p^2\psi(p, x), \tag{34}$$

$$\psi(p)|_{x=0} = 0, \tag{35}$$

$$\left[\psi(p, x) + BD_a\frac{d}{dx}\psi(p, x)\right]_{x=a} = \mu(p). \tag{36}$$

We introduce the auxiliary solutions of (34) with the boundary conditions

$$\omega(p, x=0) = 0, \tag{37}$$

$$\omega'(p, x=0) = 1. \tag{38}$$

The functions ψ and ω, being solutions of the same equation (34) and having one common boundary condition (35), (37), differ from each other only in the normalization constant, in this case $\psi'(p, x = 0)$. With allowance for this, we rewrite the boundary condition (36):

$$\psi'(p, x=0)\left[\omega(p, x) + BD_a\frac{d}{dx}\omega(p, x)\right]_{x=a} = \mu(p). \tag{39}$$

Since we take u'(0, t) as known at each time t, $\psi'(p, x = 0)$ is also known. The points of the poles $\psi'(p, x = 0)$, as is easly seen from (39), yield

$$\omega(p_m, a) + BD_a\left.\frac{d}{dx}\omega(p_m, x)\right|_{x=a} = 0. \tag{40}$$

It is obvious that the ordinary "direct" Sturm–Liouville problem is a solution of Eq. (34) with the boundary conditions (37) and (40) and a known D(x). And the determination of D(x) of Eq. (3) with the boundary conditions (37) and (40) from the levels $\{p_m\}$ and their normalizations $\{C_m^2\}$ is an inverse Sturm–Liouville problem, to which the problem (34)-(36) was reduced. The p_m are found from the known $\psi'(p, x = 0)$. The normalization constants $\{C_m^2\}$ are determined by the condition (38), from which it follows that all $C_m^2 = 1$. As is well known, the inverse Sturm–Liouville problem has a unique solution when the potential functions is reconstructed from a complete set of eigenvalues and their normalization constants or from two sets os eigenvalues (see, e.g., [9]).

Now we carry out a Liouville transformation using the substitution

$$\frac{dx}{d\xi} = \sqrt{D(x)} = q(x) \tag{41}$$

and convert from Eq. (34) to the Schrödinger equation (3) for the new variable ξ with

$$V(\xi) = q^{-1}\frac{d^2 q(\xi)}{d\xi^2}. \tag{42}$$

Using the equations of the inverse problem given at the beginning, we obtain

$$\sqrt{D(\xi)} = q(\xi) = q(0)\left[1 + \int_{-\xi}^{\xi} K(\xi, \sigma) Q(\sigma, \xi) d\sigma\right], \tag{43}$$

where the integral term is absent from $Q(\sigma, \xi)$ of (6'), and instead of Jost's solutions of Eq. (3) we now use auxiliary regular solutions of the type (8') determined by the boundary conditions (37) and (38) (see [1, 5]).

Summing up, we note that inverse spectral problems can be used extensively in problems of both electrodynamics and heat and mass exchange. In the present work we systematically analyzed two inverse problems in application to the formation of optical devices with assigned transmission characteristics.

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